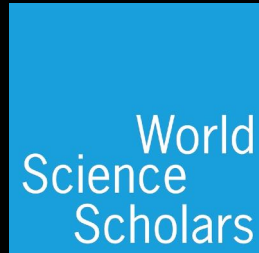


MODULE 2

A BEAUTIFUL UNIVERSE:

Black Holes, String Theory,
and the Laws of Nature as Mathematical Puzzles



Cumrun Vafa, PhD
Harvard University
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Math and physics have had a beautiful history of interaction over many millennia. The plan of my talk is to present a modern example of this link.

I present three different stories that seem at first sight unrelated, but indeed are deeply connected.

I hope this illustrates how deep ideas in science form a unified borderless enterprise.

SOME HIGHLIGHTS FROM THE PAST

Greek mathematicians and philosophers:

Math and geometry should be important in explaining nature:

Platonic solids



Represent basic
Ingredients



OCTAHEDRON
Air



CUBE
Earth



TETRAHEDRON
Fire

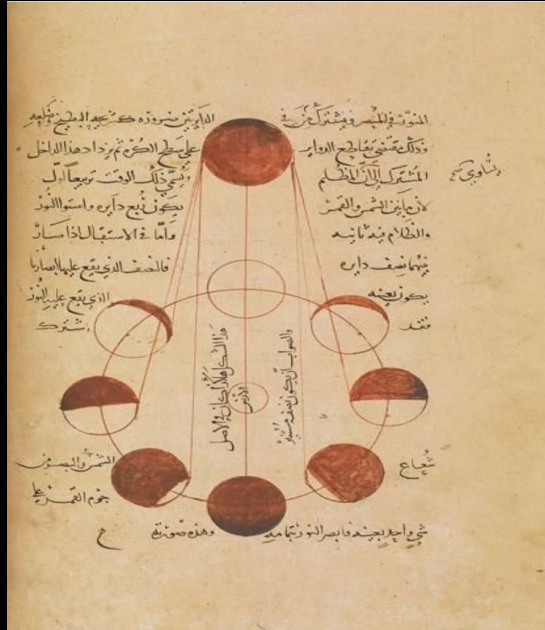


DODECAHEDRON
the Universe

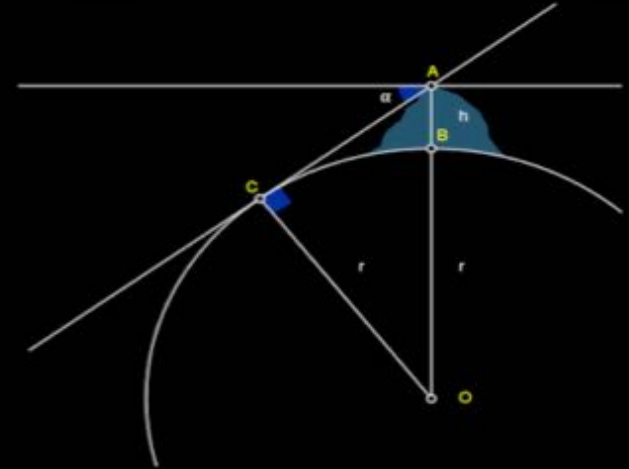


ICOSAEDRON
Water

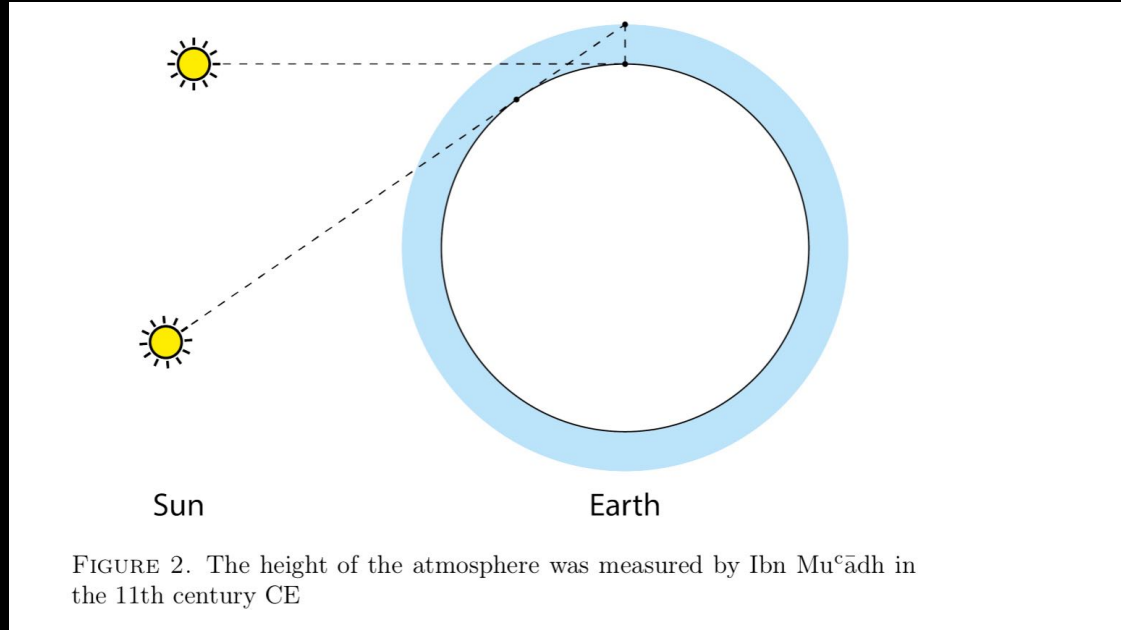
ISLAMIC SCIENTISTS AND MATHEMATICIANS CONTINUED THIS TRADITION:



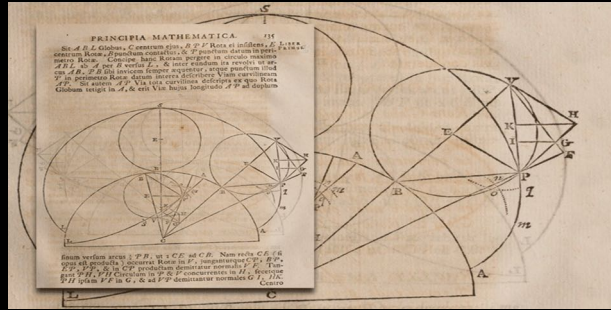
Biruni:
Accurate
measurment
of radius of
Earth



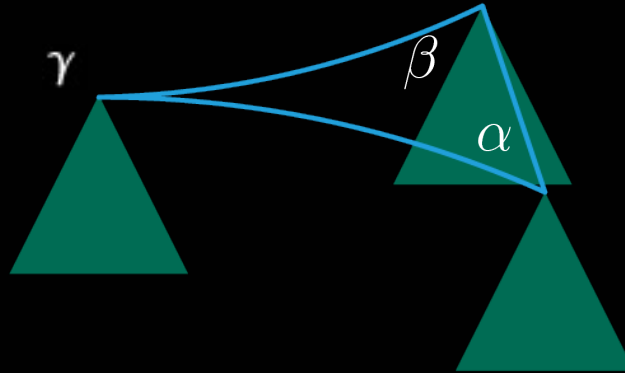
ISLAMIC SCIENTISTS AND MATHEMATICIANS CONTINUED THIS TRADITION:



NEWTON: Calculus and laws of mechanics



GAUSS: Perhaps our universe is non-Euclidean

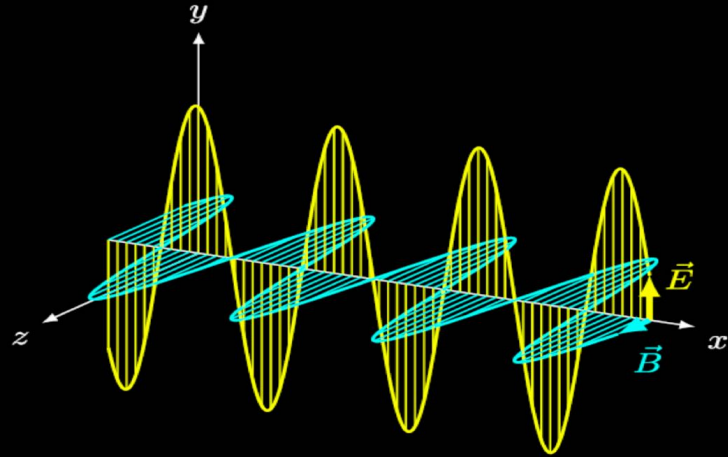


$$\alpha + \beta + \gamma = 180?$$

MAXWELL:

Electricity and magnetism captured by differential calculus

$$d^* F = * J$$



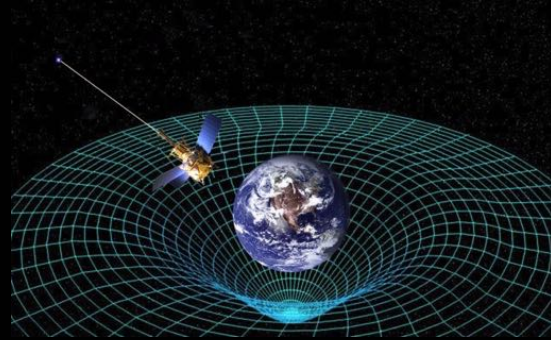
RIEMANN:

Can Riemannian geometry be used to unify electric/magnetic phenomena and gravity?

EINSTEIN:

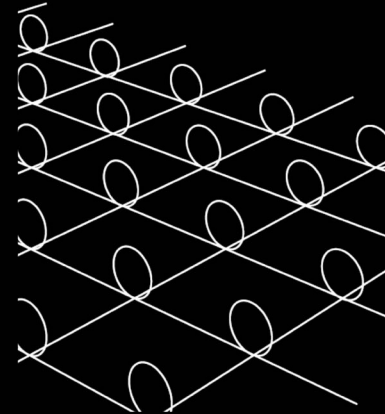
Riemannian geometry in 3+1 dimensions encodes gravitational force.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}$$



KALUZA AND KLEIN:

Einstein's theory in 4+1 dimensions unifies gravity and electromagnetism.



This brings us to the early 20th century, where our story of today starts to unfold...

SRINIVASA RAMANUJAN

1887-1920

न.पे.
15
np

INDIA POSTAGE

INDIA SECURITY PRESS

STORY I: RAMANUJAN



Among the many amazing works that Ramanujan did in his short lifetime, the following (joint work with Hardy) has proven to be rather important:

Consider an integer, say 5. A partition of 5 is to break it up as a sum of integers; the number of inequivalent ways of doing that is denoted by $P(5)$.

$$5=1+1+1+1+1, 1+1+1+2, 1+2+2, 1+1+3, 2+3, 1+4, 5$$
$$P(5)=7$$

The problem of finding $P(n)$ gets increasingly difficult as we increase $n \gg 1$.

Hardy and Ramanujan came up with an ingenious way to compute this number in closed form. For the purpose of this talk the leading behavior that they found for large n suffices. They found using properties of modular forms that

$$P(n) \propto e^{\pi \sqrt{\frac{2}{3}n}}$$

$$\sum P(n)q^n = \frac{1}{\prod (1-q^n)}$$

$$\eta(q) = q^{1/24} \prod (1-q^n)$$

$$q = e^{2\pi i \tau}$$

$$\tau \rightarrow i0 \quad q \rightarrow 1$$

$$1/\eta = q^{-\frac{1}{24}} \sum P(n) q^n$$

This gets dominated for large values of n.

$$\eta(-1/\tau) = \sqrt{-i} \cdot \eta(\tau)$$

$$\begin{aligned} -1/\tau &\rightarrow i\infty \\ q &\rightarrow 0 \end{aligned}$$

$$1/\eta \rightarrow q^{-\frac{1}{24}}$$

$$P(n) \propto e^{\pi \sqrt{\frac{2}{3}n}}$$

Their method of proof was reinterpreted 60 years after their work in physical terms (in the context of string theory), in terms of oscillations modes of a string:



For each integer **n**, there is a harmonic on the string. The higher the harmonic, the higher the energy in the string. In fact we can take the energy of the **nth** harmonic mode to be (in some unit)

$$E = n$$

Next imagine putting the string in a thermal bath, so that all the harmonics can be in principle excited with probability given by Boltzmann distribution:

$$p(E) \propto \exp(-\beta E); \beta = \frac{1}{kT}$$

Where **E** stands for the total energy of the modes that have been excited. In particular if we have a superposition of excitations of the harmonics given by

$$n_1, n_2, \dots, n_k$$

Then the energy is given by

$$E = n_1 + n_2 + \dots + n_k$$

The thermal partition function of the system is given by summing over all allowed energies in the system:

$$Z = \sum_{\{n_i\}} e^{-\beta E(\{n_i\})}$$

Note that if you fix an energy E , the number of inequivalent configurations with that energy, is exactly the number of partitions of E to integers, i.e., $P(E)$. This is because for a given E we have to find the harmonics which add up to E . So we can write:

$$Z = \sum_{E=0}^{\infty} P(E) e^{-\beta E}$$

Thus finding an asymptotic behavior for $P(E)$ for large E , boils down to computing

$$Z(\beta) \quad \text{as} \quad \beta \rightarrow 0$$

In quantum mechanics, a state with energy E evolves at time t by

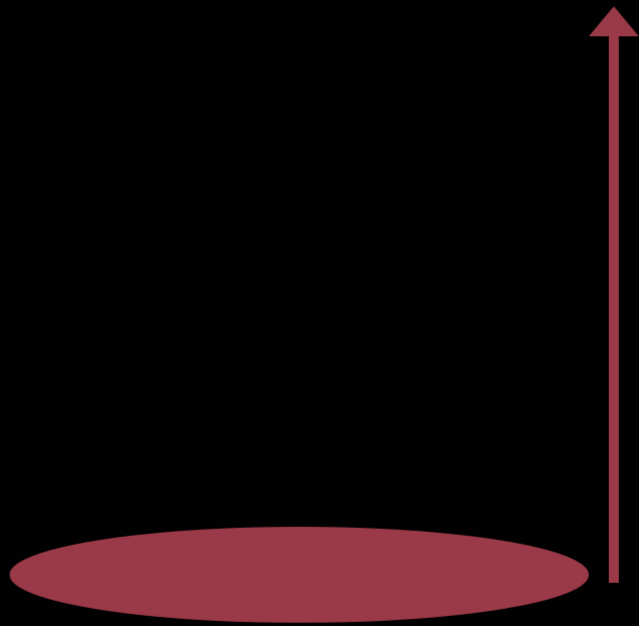
If we change variables to Euclidean time

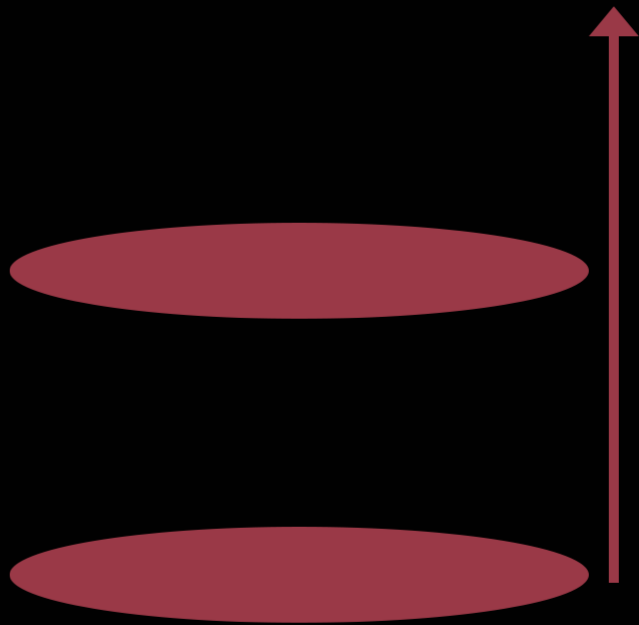
$$it \rightarrow \beta$$

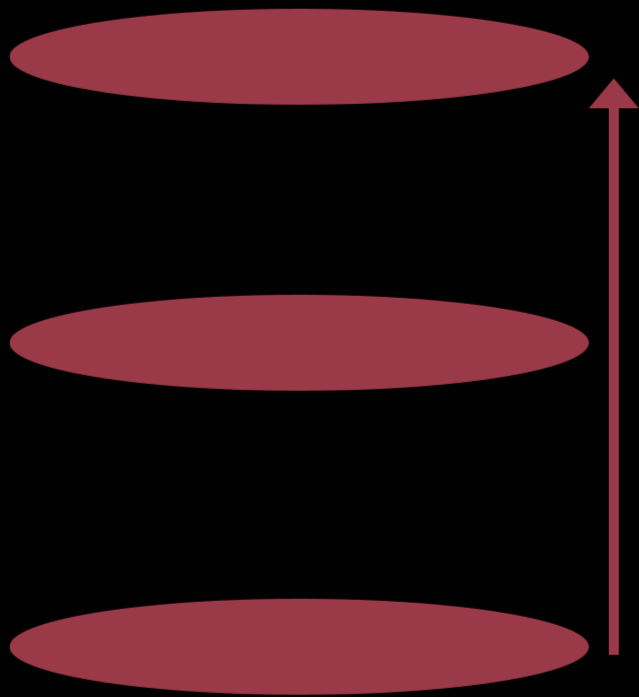
This becomes

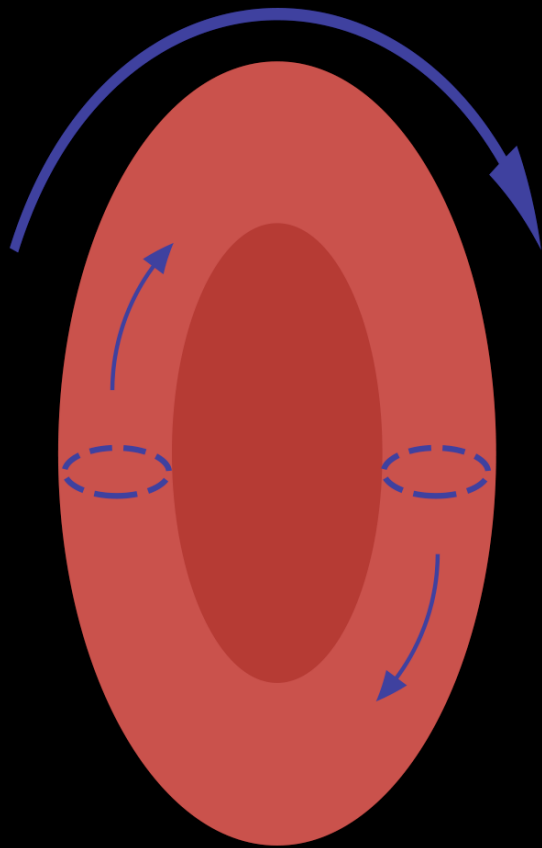
$$e^{-\beta E}$$

Which looks like Boltzmann distribution.



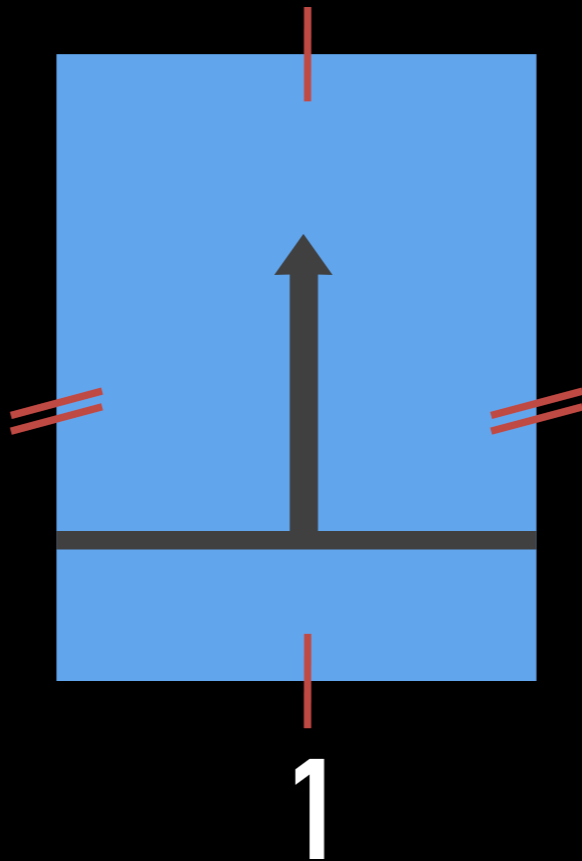




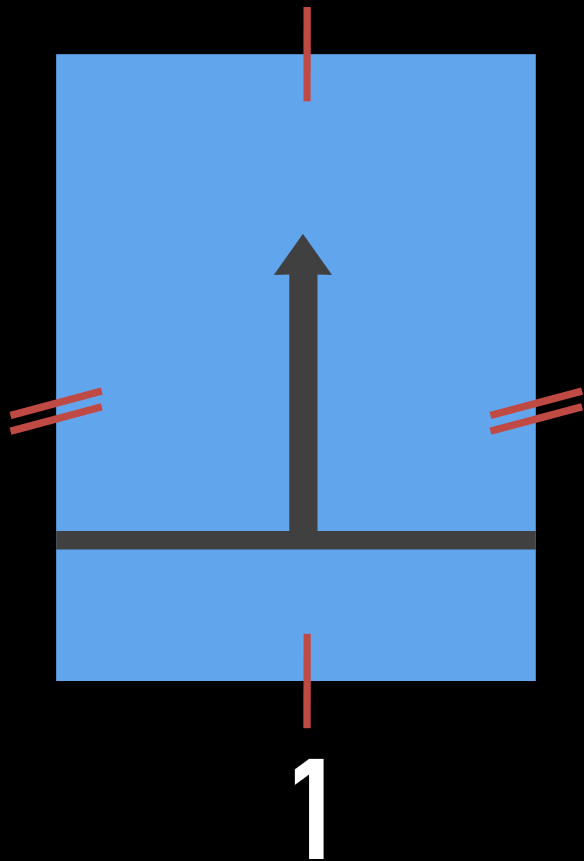


$$= Z(\beta)$$

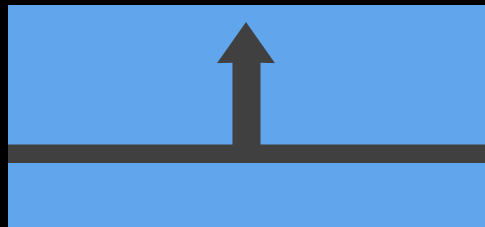
β



We are interested in
the limit of small

 β β 

β



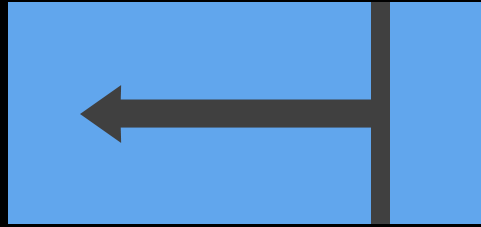
1

β



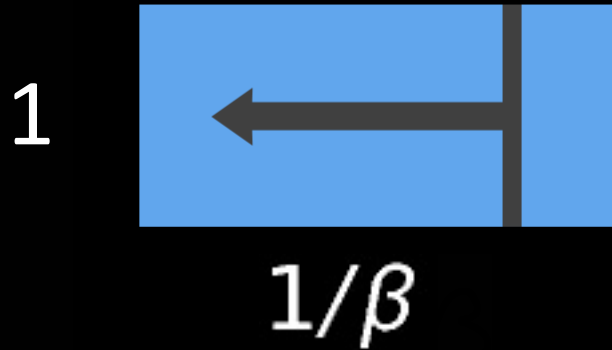
1

The partition function is scale invariant, so we can rescale both sides by the same amount without changing Z

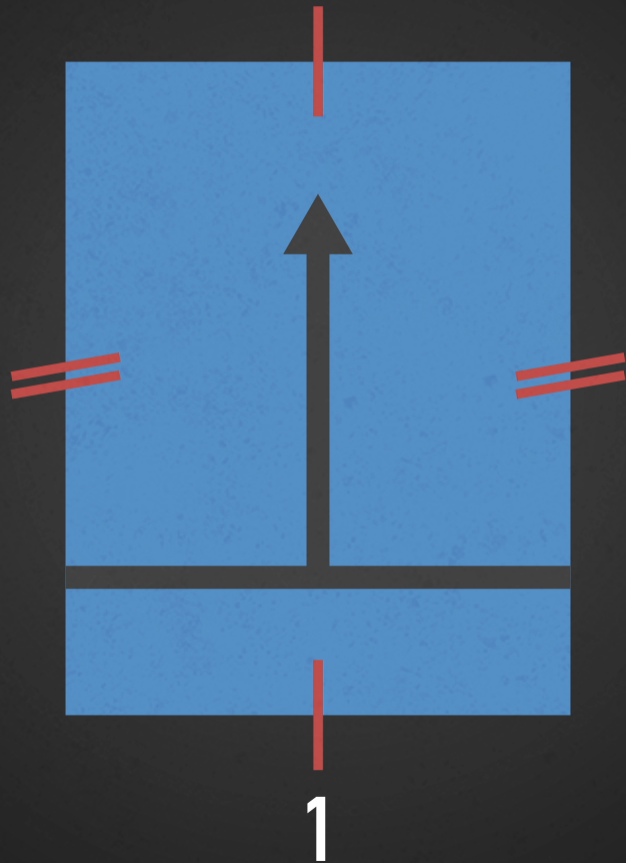
 β 

1

The partition function is scale invariant, so we can rescale both sides by the same amount without changing Z



β





Recalling that we are interested in small β , this inversion symmetry implies that we might as well compute the partition function for large values of β which is dominated by smallest value of E

$$\exp\left(\frac{-E}{\beta}\right)$$

i.e. Without any harmonic excitations: the ground state. Classically this energy is 0. But quantum mechanically it gets a small negative correction and the ground state turns out to be

Leading to:

$$Z(\beta) = e^{\frac{\pi^2}{6\beta}} \quad E = \frac{-(2\pi)^2}{24}$$

From which one can deduce that

$$P(E) \sim e^{\pi\sqrt{\frac{2E}{3}}}$$

$$E = \frac{1}{2}(2\pi)^2 \sum_{n=1}^{\infty} n = -\frac{(2\pi)^2}{24}$$

This is based on mathematical idea of ‘analytic continuation’:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \rightarrow \zeta(-1) = -\frac{1}{12}$$

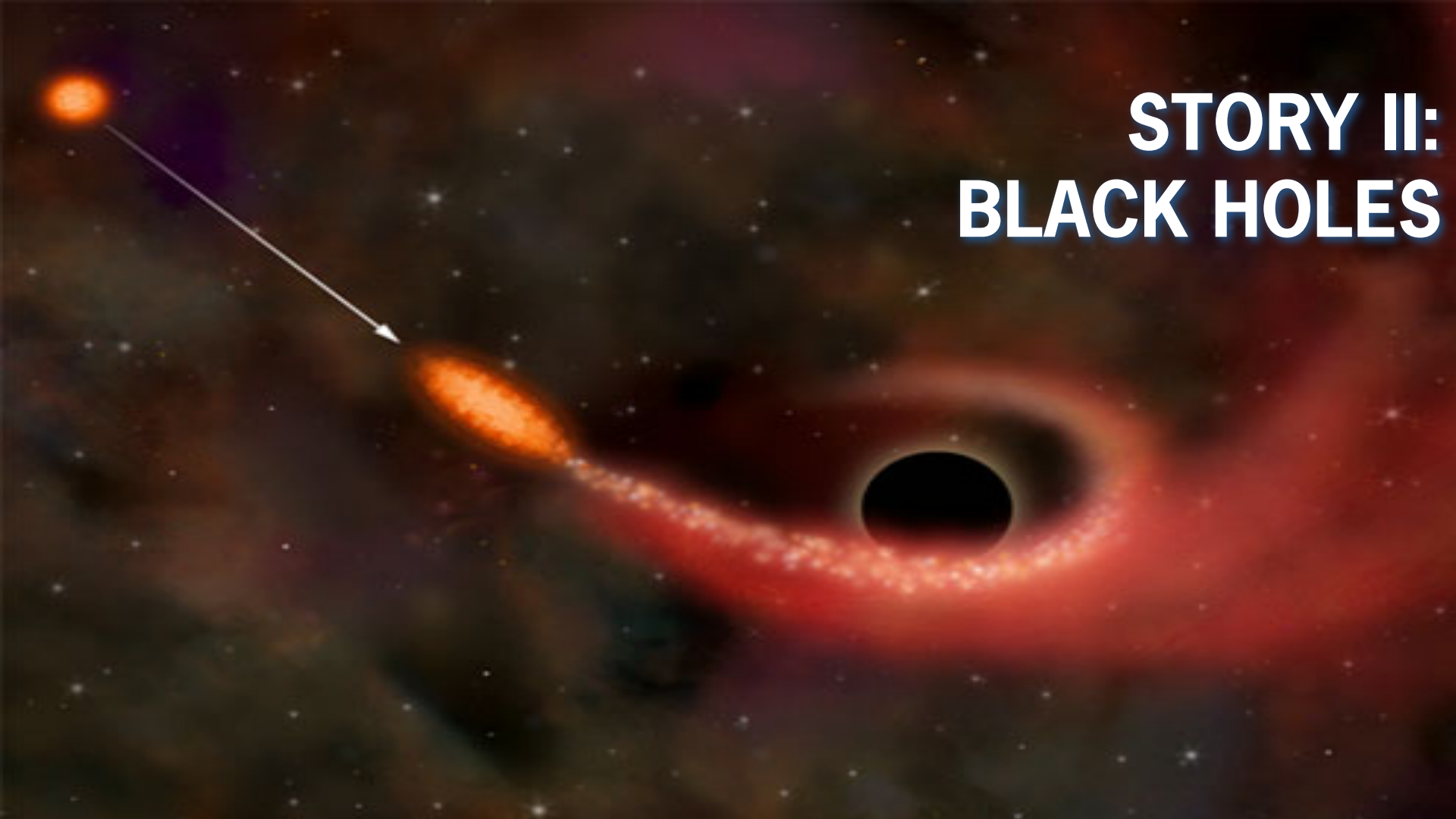
(As an exercise using geometric series show
 $1+2+4+8+16+\dots = -1$)

$$P(E) \sim e^{\pi \sqrt{\frac{2E}{3}}}$$



$$P(n) \sim e^{\pi \sqrt{\frac{2n}{3}}}$$

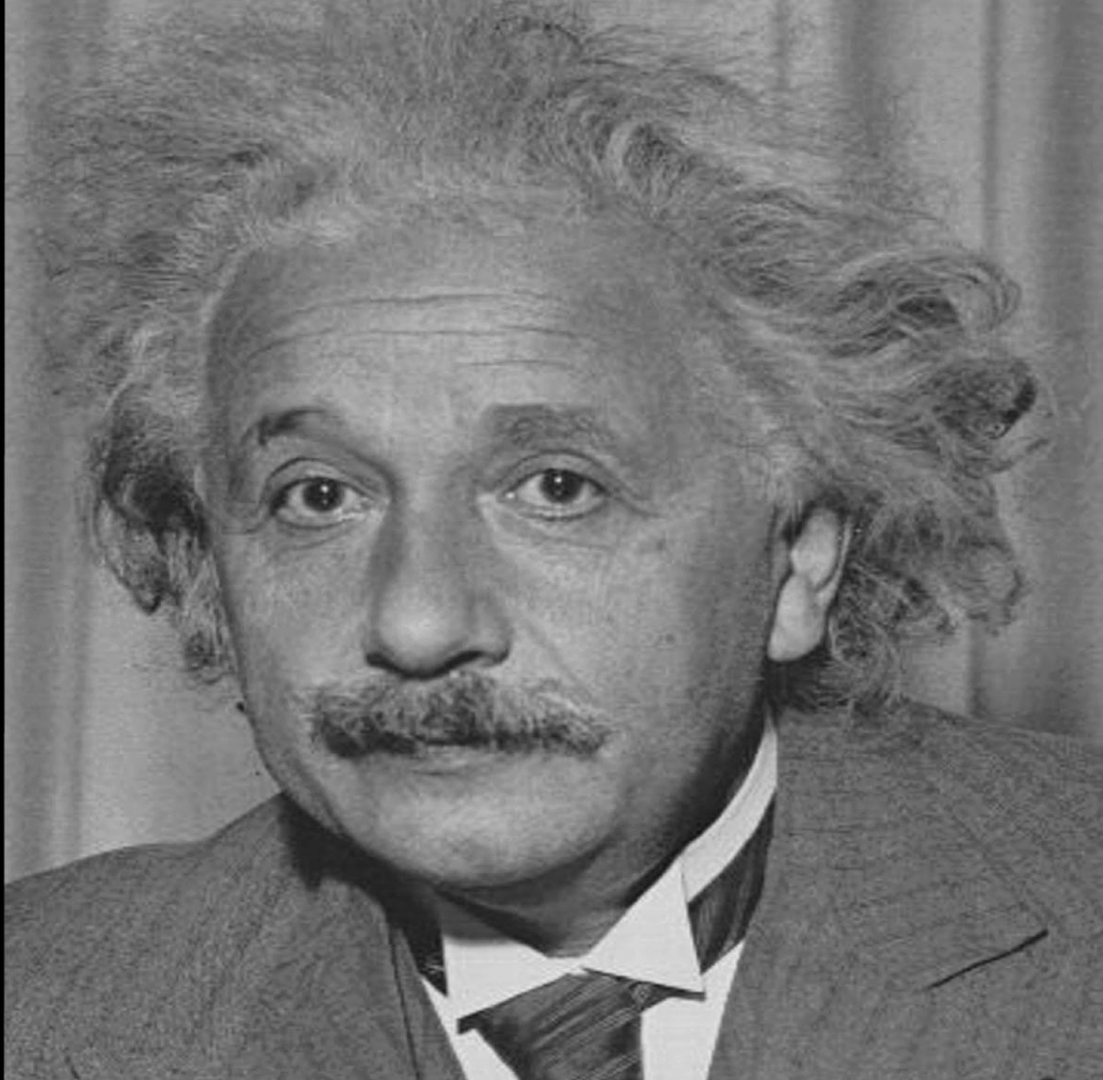
STORY II: BLACK HOLES



NEWTON
1642-1727



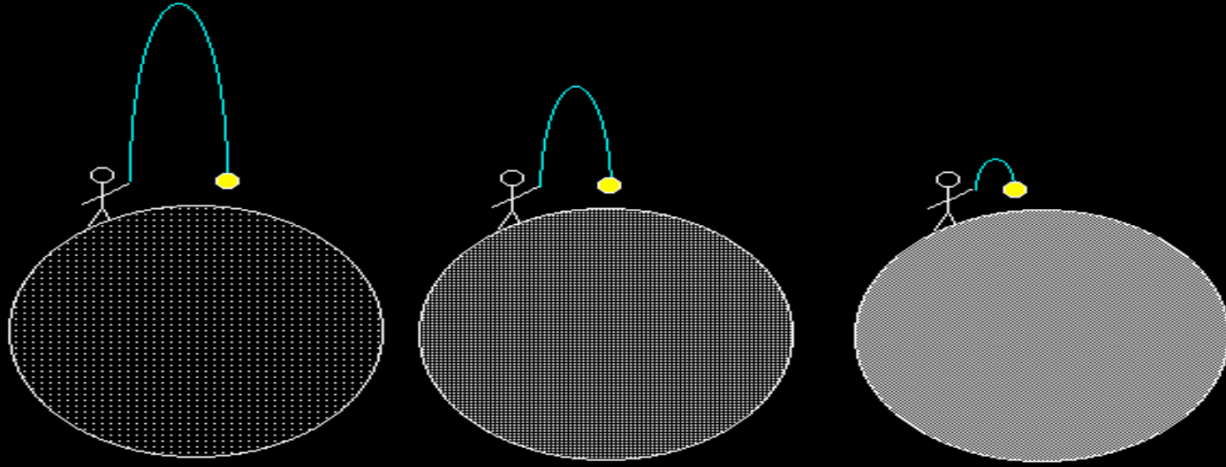
EINSTEIN
1879-1955



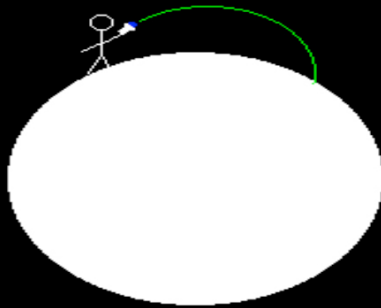
**STEPHEN
HAWKING**
1942-2018



As the mass increases, so does the gravitational pull



If the gravitational pull is such that even light cannot escape, then a black hole forms



LIGHT RAY

$$\frac{1}{2}mv^2 - \frac{GmM}{R} = 0$$

BLACK HOLE

$$R = \frac{2GM}{c^2}$$





Black holes are among the most enigmatic objects in the universe. They have been a source of inspiration for physicists ever since their discovery. They were so mysterious that even Einstein had difficulty believing that they can even exist! In fact, the laws of physics seem to break down once you fall into the black hole: After a short time inside a black hole you hit a singularity, beyond which the current known laws of physics break down. It has been one of the most challenging problems of theoretical physics to find out what happens after this, and what replaces the current known laws of physics after you hit this singularity.

No one currently knows the answer to this question.

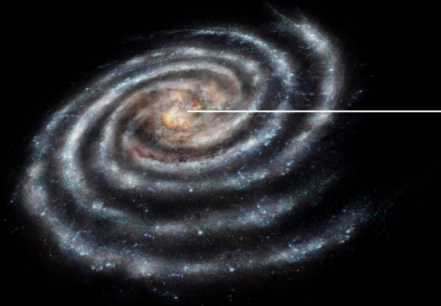
TYPES AND SIZES OF BLACK HOLES



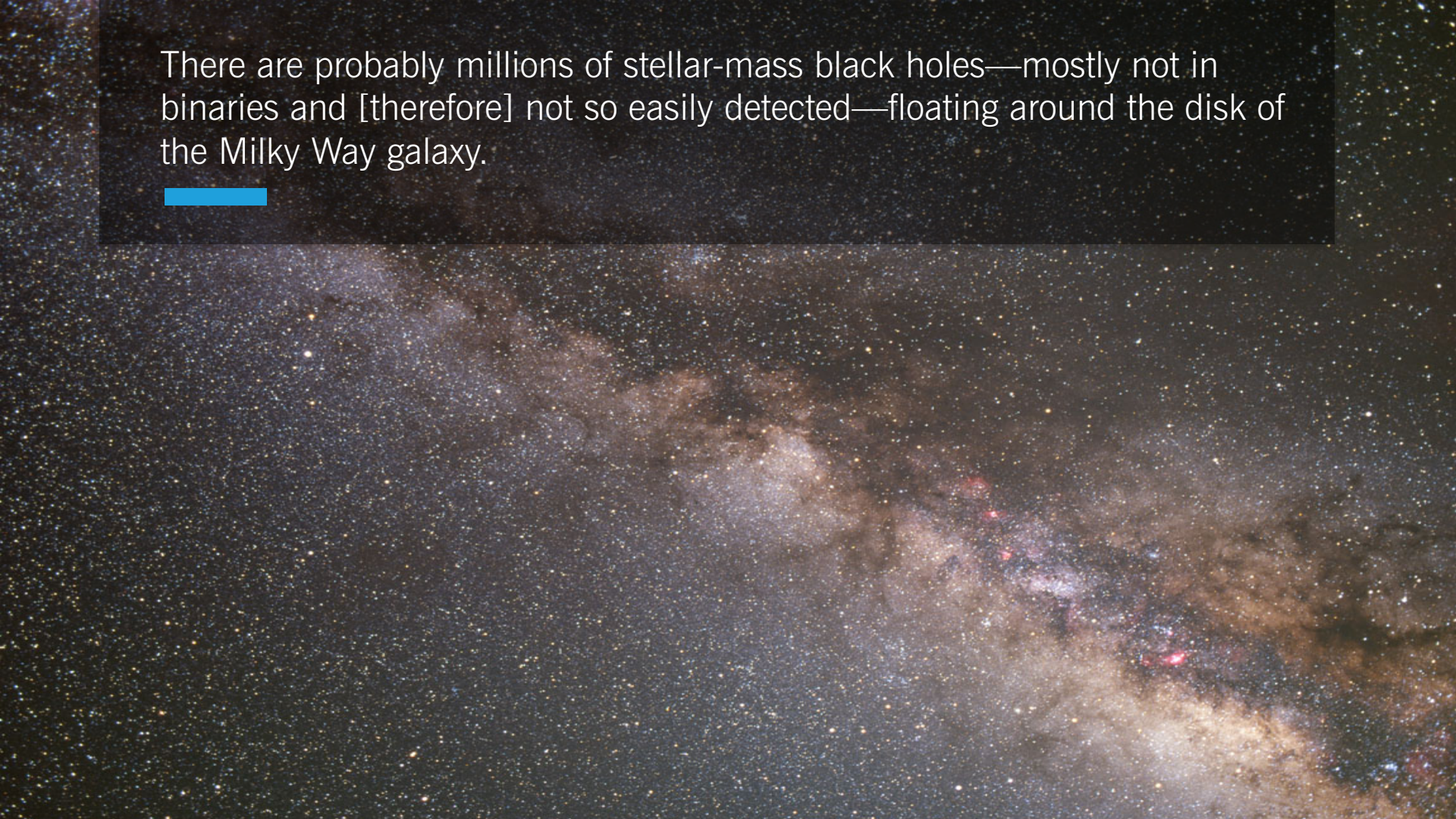
A black hole is classified by the only three properties that it possesses: mass, spin, and charge.

Currently, there are only two recognized mass classes of black hole: stellar and supermassive. The stellar black holes are star-sized and range in the 10-100 solar mass range. The supermassive black holes are at the cores of—what appear to be—every large galaxy, including our Milky Way. These range in the millions to even billions of solar masses.

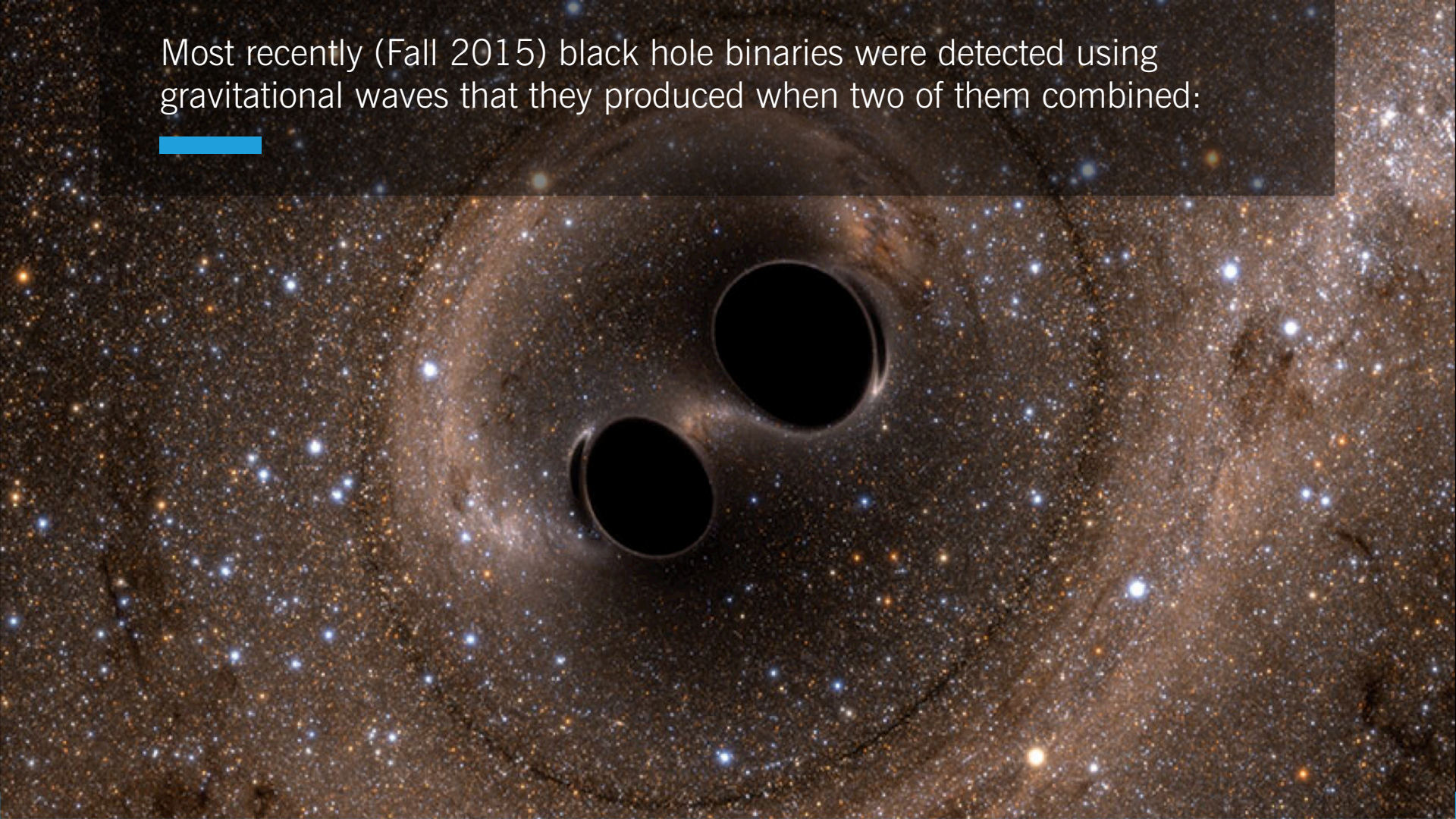
One massive black hole candidate we are certain about lies at the heart of our own galaxy. This black hole is as massive as about three million suns dwelling about 25,000 light years from Earth.

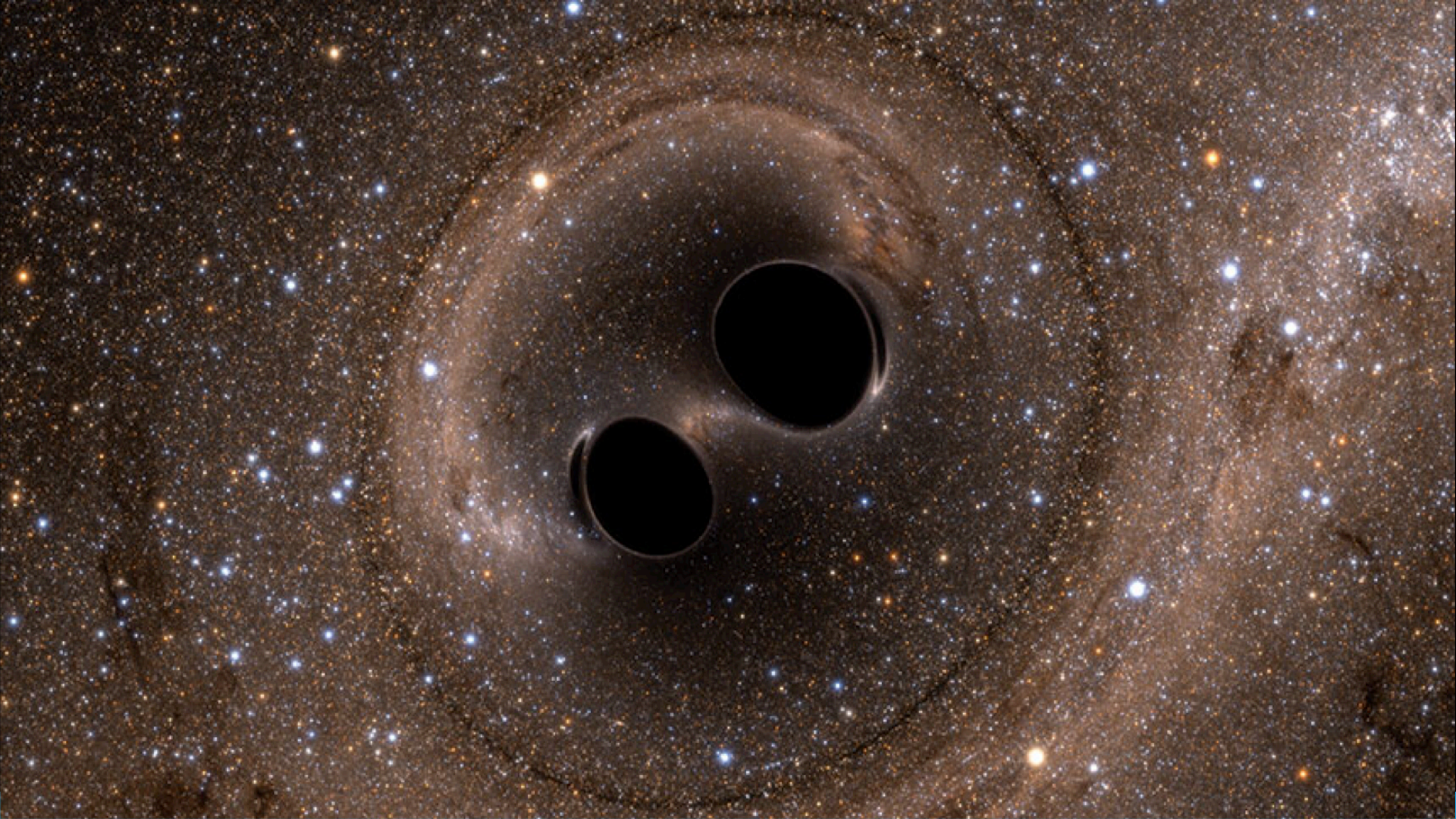


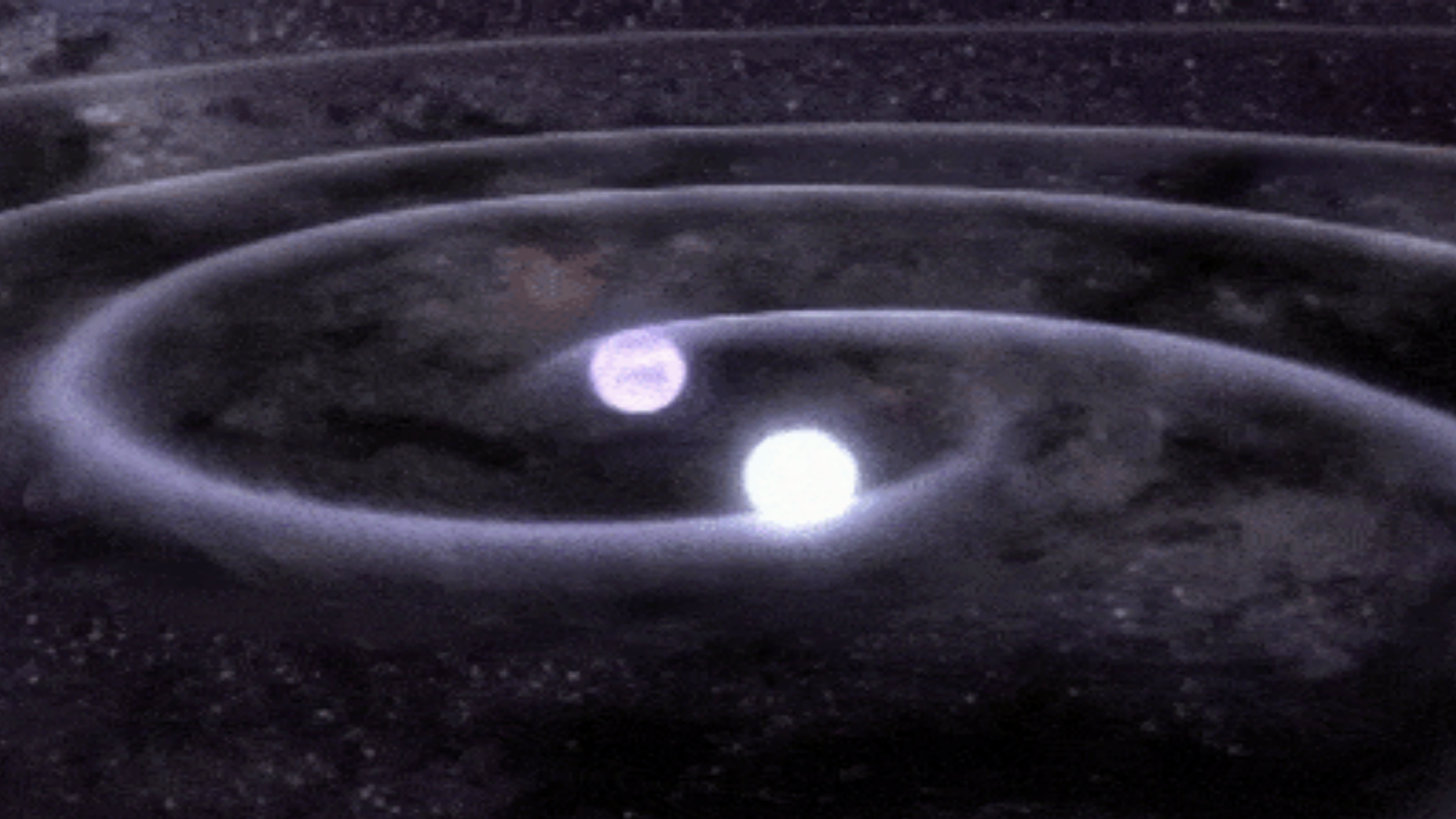
There are probably millions of stellar-mass black holes—mostly not in binaries and [therefore] not so easily detected—floating around the disk of the Milky Way galaxy.

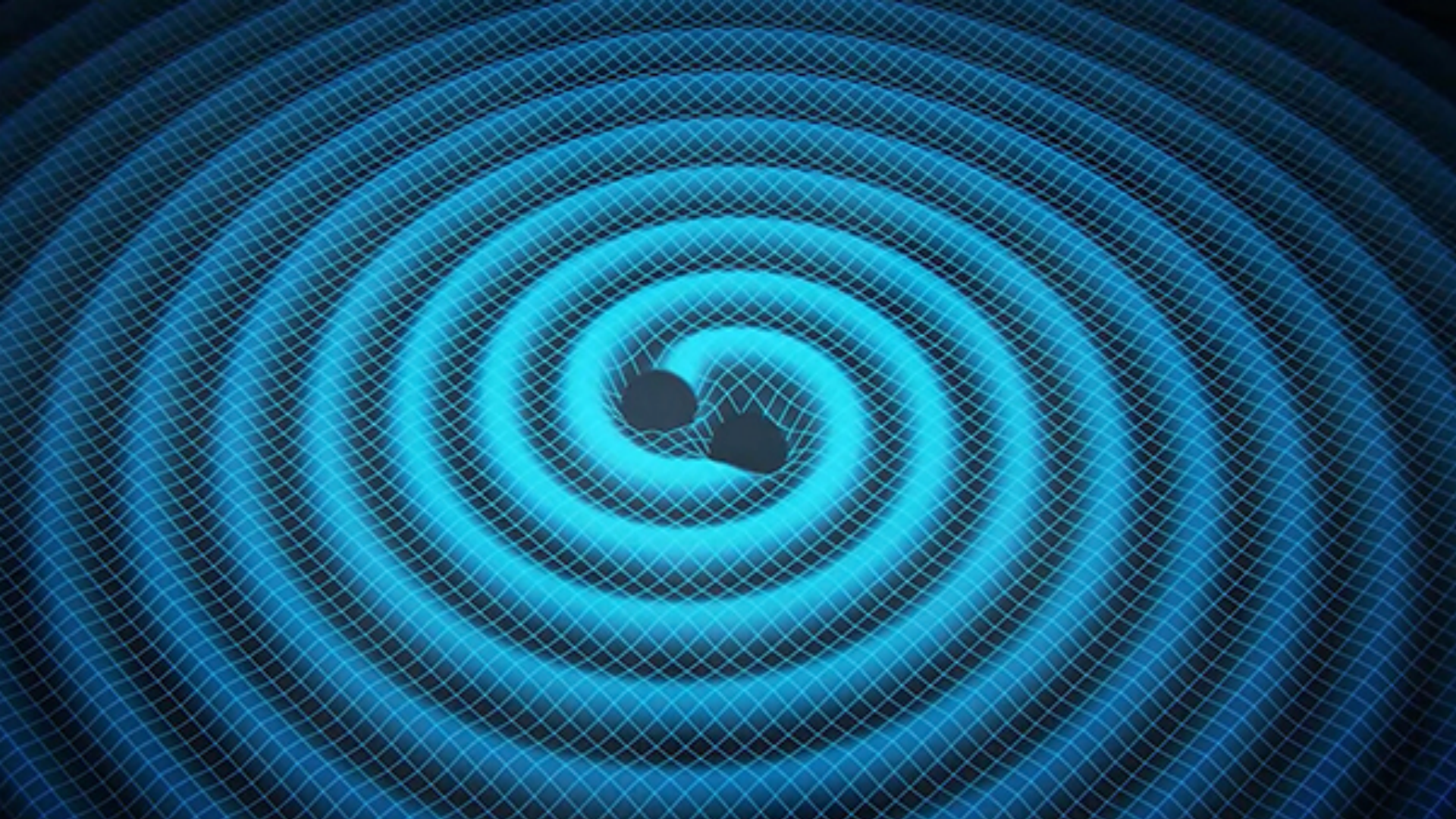


Most recently (Fall 2015) black hole binaries were detected using gravitational waves that they produced when two of them combined:



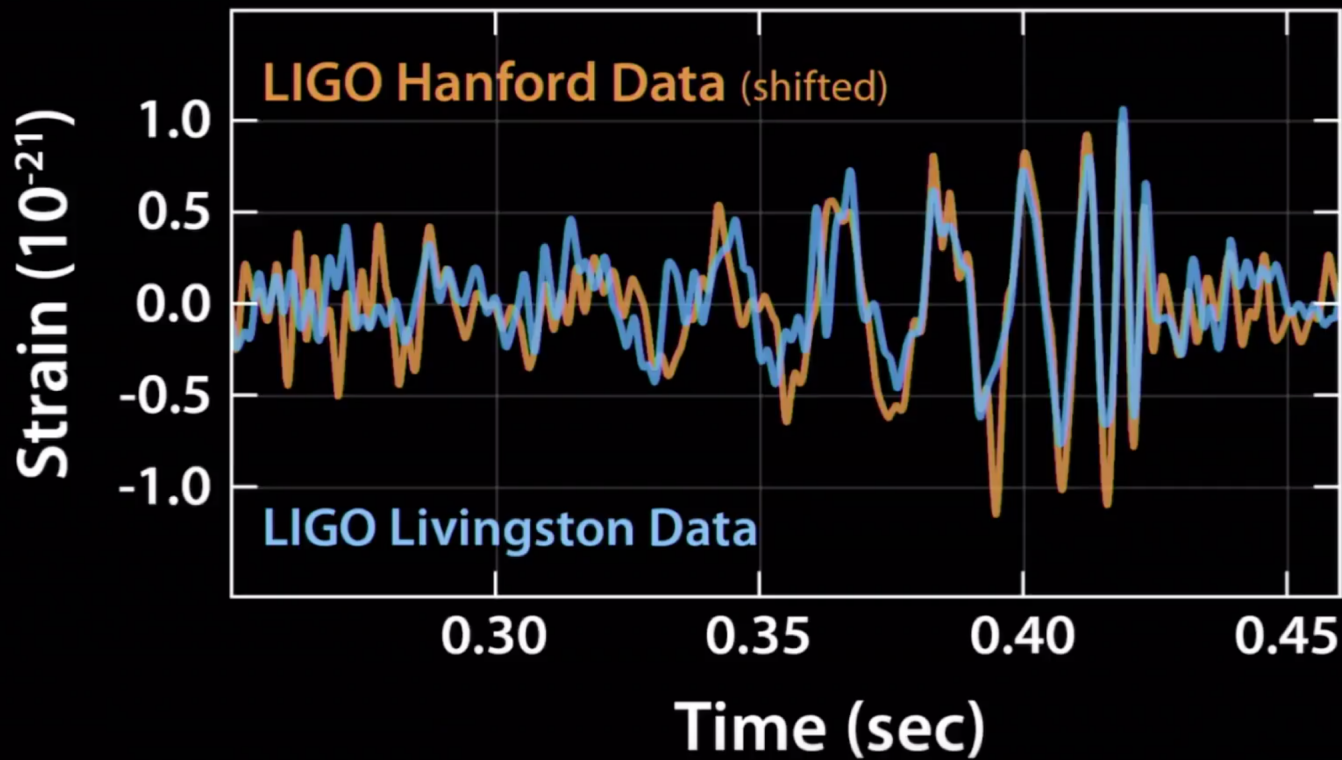


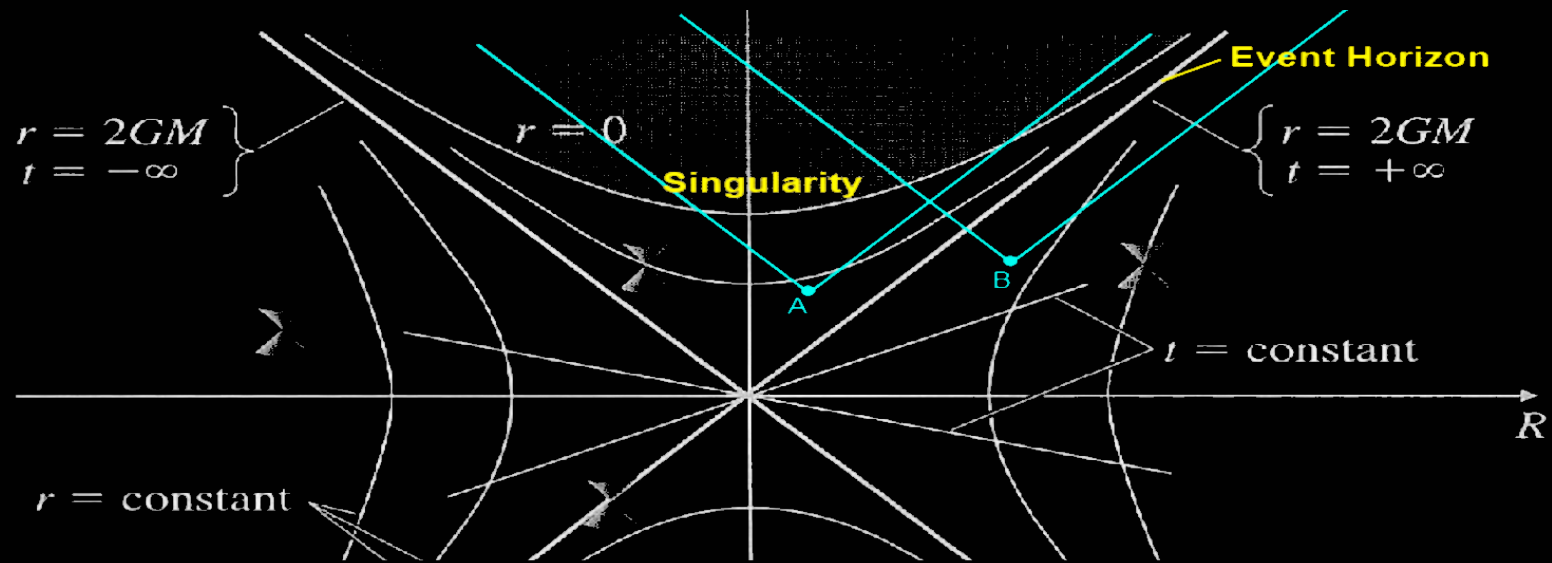






LIGO EXPERIMENT





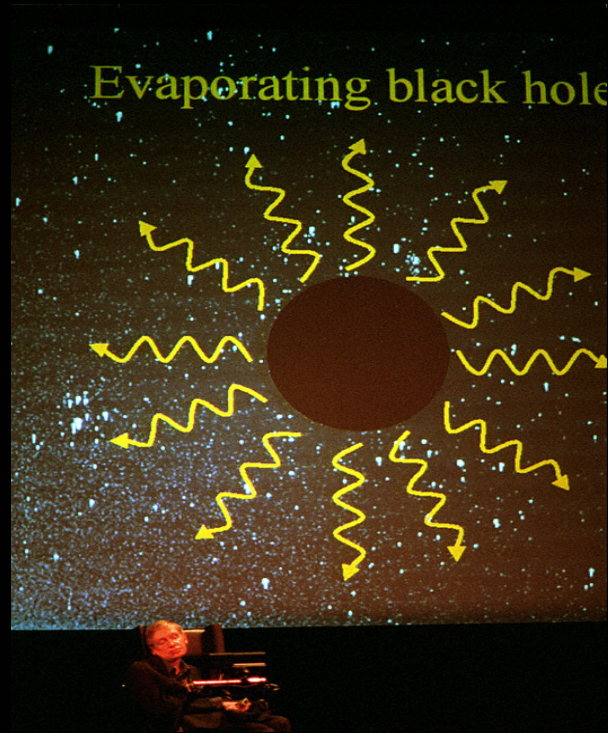
So what is the fate of black holes once they are formed? Won't they last forever and eat up all the matter?

Hint: When black holes merge areas are not additive.

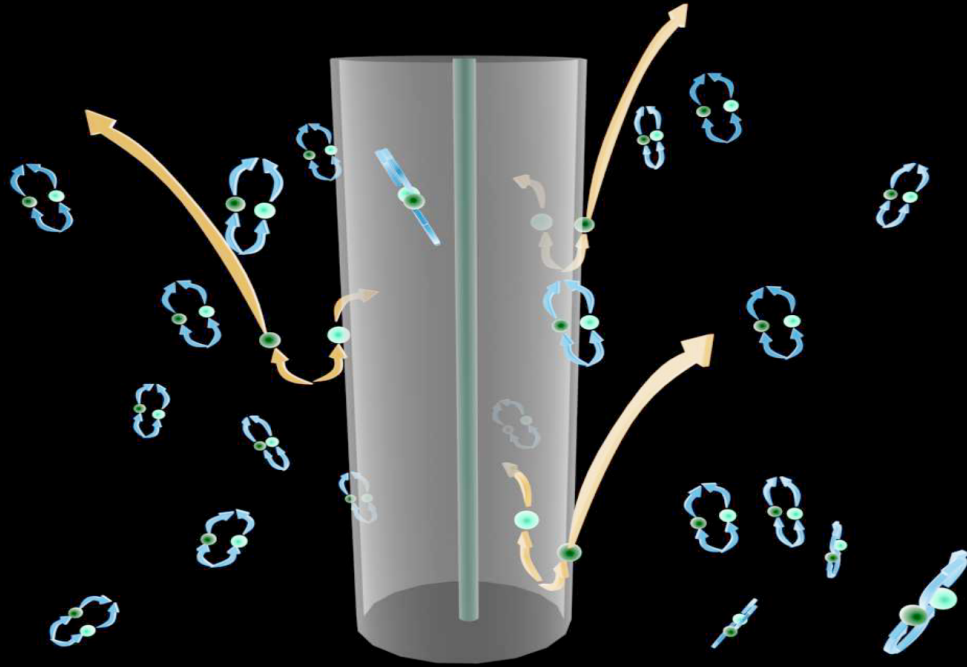
$$A_{12} > A_1 + A_2$$

Bekenstein: Could it be that A is proportional entropy? If so, what proportionality?

Stephen Hawking has found the answer:



Black holes are not quite black. They do radiate light outwards due to a quantum effect.



HAWKING RADIATION

But, there is a puzzle:

For an object like a black hole to radiate and have thermodynamic properties, like temperature, it should have a huge number of microstates. However, Einstein's equations predict that the solution for a given black hole mass is unique, so it has no microscopic degrees of freedom.

At least for classical black holes.

Hawking, building on the work of Bekenstein, found that consistency of quantum effects with black hole physics suggests there must be microstates whose number is

$$N \sim e^{\frac{A}{4}}$$

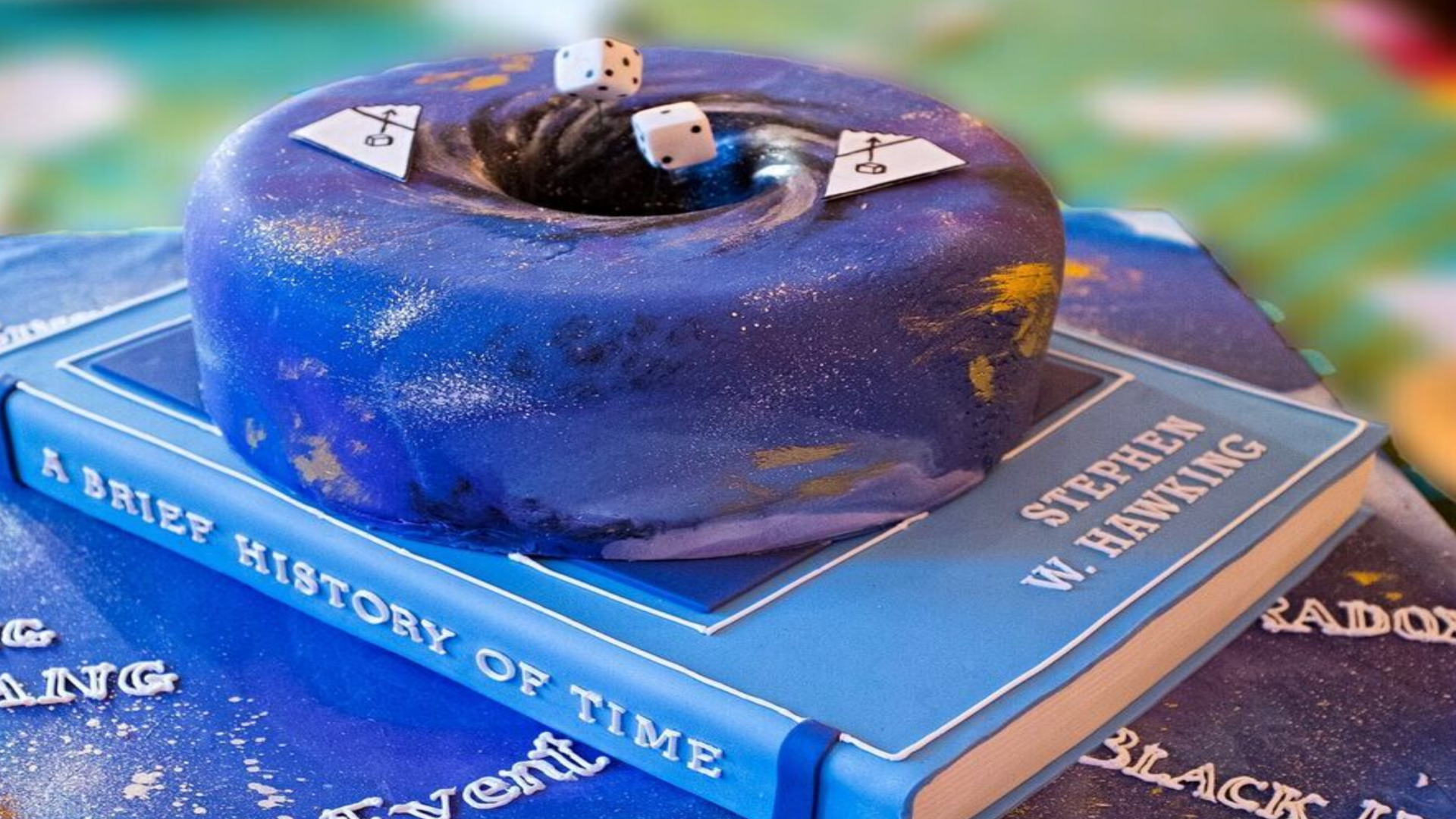
Here A = area of horizon

But, he could not identify what these states are!

$$S = \frac{A}{4} \quad A = 8\pi M^2$$

$$dE = T dS \quad E = M$$

$$T = \frac{1}{4\pi M}$$



A BRIEF HISTORY OF TIME

STEPHEN
W. HAWKING

ANG

RADOX

BLACK